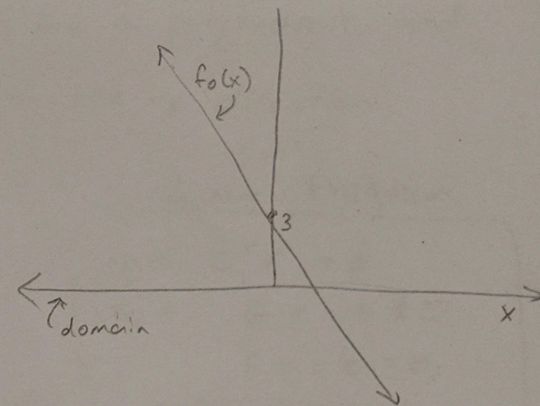
(1) $x \in \mathbb{R}$, $f_0(x) = -2x + 3$

What kind of function is this?



$$\min_{x \in \mathbb{R}} \; -2x + 3$$

what x would we choose to find the minimum value p^*?

From inspection, $x^* = \infty$, $p^* = -\infty$

Let's look at the more general problem:

(2) $f_0(x) = cx + d$, $c, x, d \in \mathbb{R}$

How would we look for x^*? Does it depend on d at all?

if $c > 0$, $x^* = -\infty$, $p^* = c(-\infty) + d = -\infty$

if $c < 0$, $x^* = \infty$, $p^* = c(\infty) + d = -\infty$

if $c = 0$, what happens? $p^* = cx + d = 0(x) + d = d$

So if c is nonzero, we always choose x^* to be as far as possible in the $(-c)$ direction. Does this match with our earlier problem (1)?

What happens if we add a constraint, for example $x \leq 5$?

(3) Let's move beyond scalars now.

$$\min_{x_1, x_2} \; [-1 \;\; 3] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 3 \quad \longleftrightarrow \quad \min_{x} \; c^T x + d$$

where $c = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $d = 3$

Show matlab figures, level sets

$\nabla f_0(x) = c = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. To move in directions of negative gradient (i.e. gradient descent), we move in direction of $-c$

(4) Add constraints

$$\min_{x_1, x_2} \; [-1 \;\; 3] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 3 \quad \longleftrightarrow \quad \min \; c^T x + d$$

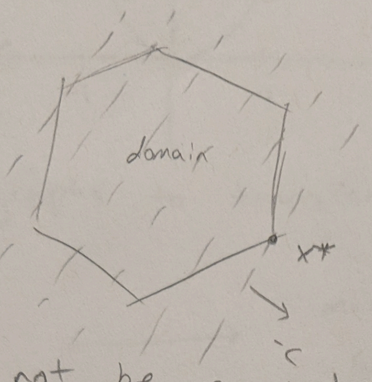
s.t. $-2 \leq x_1 \leq 2$
$-2 \leq x_2 \leq 2$

s.t. $\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \leq$

So in general, our feasible set will be a polyhedron, and x^* will be the x as far as possible in the $-c$ direction (i.e. most negative value of c^Tx)

Linear Program

$$\boxed{\begin{aligned} &\min\ c^Tx + d \\ &\text{s.t. } Gx - h \le 0 \\ &\qquad Ax - b = 0 \end{aligned}}$$



Will x^* ever not be a vertex of this domain?

So why not just check all vertices?

# of vertices scales dramatically w/ constraints

LP examples

- Alex's wine making business

maximize profit

s.t. can't produce negative bottles

can't use more grapes than I have

Merlot: 2000 kg

Shiraz: 1500 kg

Blends	# Merlot	# Shiraz	Profit
1	10	5	\$20
2	8	4	\$15
3	7	15	\$25

maximize $\$20q_1 + \$15q_2 + \$25q_3$

s.t. $q_1, q_2, q_3 \ge 0$

$10q_1 + 8q_2 + 7q_3 \le 2000$

$5q_1 + 4q_2 + 15q_3 \le 1500$

$\longleftrightarrow$ $\min\ c^Tx$ s.t. $Gx - h \le 0$

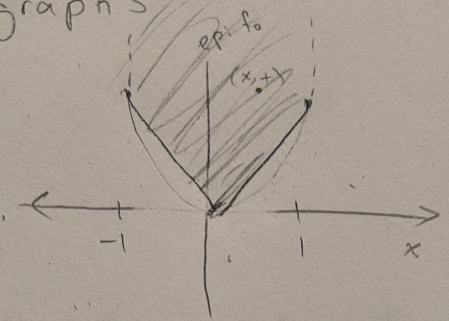
$$c = -\begin{bmatrix} 20 \\ 15 \\ 25 \end{bmatrix} \quad x = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \quad \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 10 & 8 & 7 \\ 5 & 4 & 15 \end{bmatrix}}_{G} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} - \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 2000 \\ 1500 \end{bmatrix}}_{h} \le 0$$

Can we turn nonlinear problems into LP's?

First, recall epigraphs

min $f_0 = |x|$

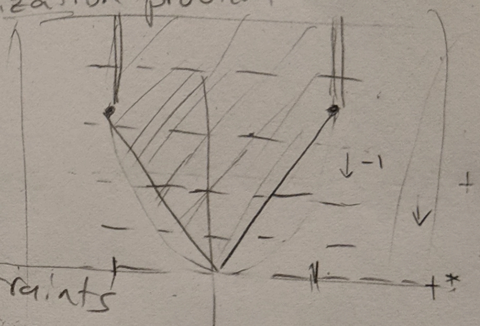
s.t. $-1 \le x \le 1$



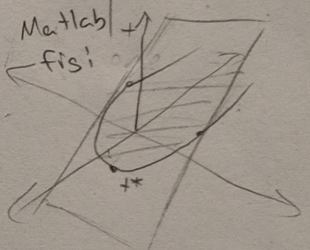
$\text{epi } f_0 = \{(x,t) \in \mathbb{R}^{n+1} \mid x \in \text{dom} f, \ f(x) \le t\}$

We can use epigraphs to transform our optimization problem

min $|x|$ s.t. $-1 \le x \le 1$ $\longleftrightarrow$ min t s.t. $x \le t$, $x \ge -t$, $-1 \le x \le 1$

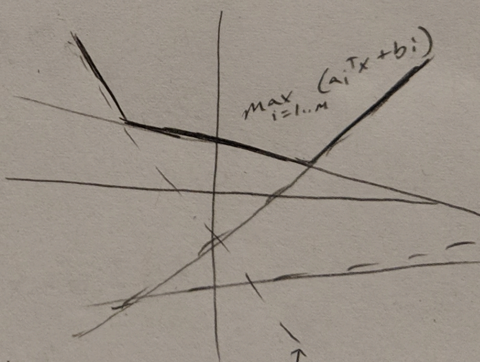


- Is this now an LP? Yes! Affine objective, affine constraints!
- When does this help us turn problems into LP's? (insert example that doesn't work: x^2)
- Need non-affine objectives to turn into affine constraints.



Example: piecewise-linear minimization

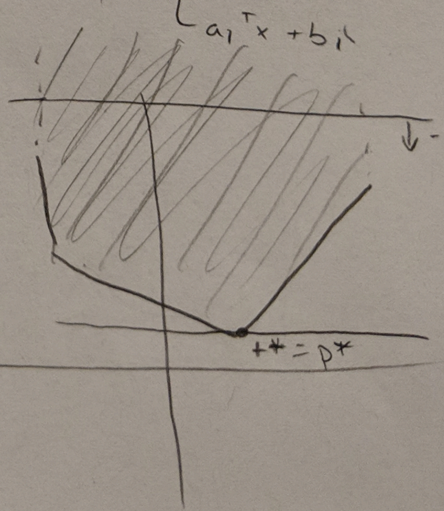
minimize $\max_{i=1\ldots m} (a_i^T x + b_i)$



is this convex? yes

is this affine? No! Not a line at all.

However, if we do our epigraph trick...



minimize t

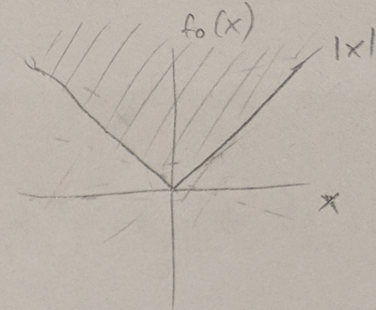
s.t. $a_i^T x + b_i \le t$, $i = 1 \ldots m$

affine objective, affine constraints.

We say that a function is a polyhedral if its epigraph is a polyhedron. Polyhedral functions can be turned into LP's

Other polyhedral functions:

$$f(x) = \|x\|_\infty = \max_{i=1\ldots n} |x_i| = \max_{i=1\ldots n} \max(x_i, -x_i)$$



$$f(x) = \|x\|_1 = \sum_{i=1\ldots n} |x_i| = \sum_{i=1\ldots n} \max(x_i, -x_i)$$

General form for minimizing polyhedral functions:

$$\min_x f(x) \quad \text{s.t. } Ax \preceq b$$

$$\updownarrow$$

$$\min_{x,t} t \quad \text{s.t. } Ax \preceq b,\ (x,t) \in \text{epi } f$$

---

l_∞ regression problems

$$\min_x \|Ax - b\|_\infty, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m$$

rewrite with epigraph form:

$$\min_{x,t} t \quad \text{s.t. } \|Ax-b\|_\infty \leq t$$

recall for $\|\cdot\|_\infty$, $\|Ax-b\|_\infty \longleftrightarrow \max_{i=1\ldots m} |a_i^T x - b_i| \leq t$

$$\longleftrightarrow |a_i^T x - b_i| \leq t;\ i=1,\ldots,m$$

$$\longleftrightarrow \begin{array}{l} a_i^T x - b_i \leq t,\ i=1\ldots m \\ a_i^T x - b_i \geq -t;\ i=1\ldots m \end{array}$$

Therefore, we can write our l_∞ regression problem as:

$$\begin{array}{ll} \min & t \\ \text{s.t.} & a_i^T x - b_i \leq t,\ i=1\ldots m \\ & a_i^T x - b_i \geq -t,\ i=1\ldots m \end{array}$$

ℓ_1 regression problems

$$\min_x \|Ax - b\|_1, \quad A \in \mathbb{R}^{m \times n}, \; b \in \mathbb{R}^m$$

recall that $\|x\|_1 = \sum_{i=1} |x_i|$

$$\min_x \|Ax - b\|_1 \longleftrightarrow \min_{x,t} \sum_{i=1}^m t_i \quad \text{s.t.} \quad |a_i^T x - b_i| \le t_i, \; i = 1 \dots m$$

$$\longleftrightarrow \min_{x,t} \; 1^T t$$

$$\text{s.t.} \quad a_i^T x - b_i \le t_i$$
$$a_i^T x - b_i \ge -t_i$$

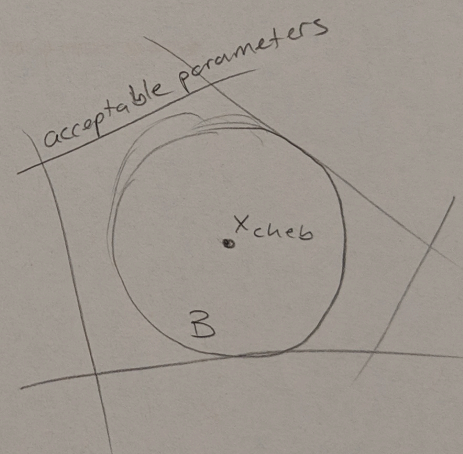
There are lots of other examples, can review LP lecture slides for examples in all different fields.

One more for us:

Chebyshev center of a polyhedron

find chebyshev center of P,

$$P = \{x \mid a_i^T x \le b_i, \; i = 1 \dots m\}$$



It is the center of largest inscribed ball

$$B = \{x_c + u \mid \|u\|_2 \le r\}$$

The constraints $a_i^T x \le b_i$ hold for all $x \in B$ if and only if

$$\sup \{a_i^T (x_c + u) \mid \|u\|_2 \le r\} \le b_i$$

$$\Updownarrow$$

$$a_i^T x_c + r\|a_i\|_2 \le b_i$$

So we can rewrite our optimization problem as

$$\text{maximize} \quad r$$
$$\text{s.t.} \quad a_i^T x_c + r\|a_i\|_2 \le b_i, \quad i = 1 \dots m$$

is this linear? What about $\|\cdot\|_2$ norm? not applied to our variable!