Constructing Control Lyapunov-Value Functions using Hamilton-Jacobi Reachability Analysis

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***Abstract*— In this paper, we seek to build connections between control Lyapunov functions (CLFs) and Hamilton-Jacobi (HJ) reachability analysis. CLFs have been used extensively in the control community for synthesizing stabilizing feedback controllers. However, there is no systematic way to construct CLFs for general nonlinear systems and the problem can become more complex with input constraints. HJ reachability is a formal method that can be used to guarantee safety or reachability for general nonlinear systems with input constraints. The main drawback is the well-known "curse of dimensionality." In this paper we modify HJ reachability to construct what we call a control Lyapunov-Value Function (CLVF) which can be used to find and stabilize to the smallest control invariant set ($\mathcal{I}_m$) around a point of interest. We prove that the CLVF is the viscosity solution to a modified HJ variational inequality (VI), and can be computed numerically, during which the input constraints and exponential decay rate γ are incorporated. This process identifies the region of exponential stability to $\mathcal{I}_m$ given the desired input bounds and γ. Finally, a feasibility-guaranteed quadratic program (QP) is proposed for online implementation.**

I. Introduction

Autonomous systems performing tasks in the real world should be both *live* (able to complete tasks) and *safe*. Control Lyapunov functions (CLFs) are a popular method to ensure liveness by stabilizing trajectories of a system to an equilibrium point [1]–[3]. Control barrier functions (CBFs) on the other hand are used to guarantee safety by maintaining trajectories of a system within a safe control invariant set [4]–[6]. Unfortunately, finding CLFs and CBFs is difficult: there lacks universal construction methods that work for general nonlinear systems. Hand-designed or application-specific CLFs and CBFs can be used [7]–[11]. However, these hand-crafted functions can be conservative in many cases, and may be invalid when faced with input bounds.

Liveness and safety can also be achieved by formal methods such as Hamilton-Jacobi (HJ) reachability analysis [12]–[16]. This method computes a value function whose level sets provide information about safety (liveness) over space and time, and whose gradients provide the safety (liveness) controller. This value function can be computed numerically using dynamic programming, handles general nonlinear systems, and can accommodate input and disturbance bounds. Undermining the appealing benefits is the "curse of dimensionality." Ongoing research has improved computational efficiency [17]–[19], but performing dynamic programming in high dimensions (6D or more) remains challenging. Additionally, standard HJ analysis does not provide the ability for a system to stabilize to a goal.

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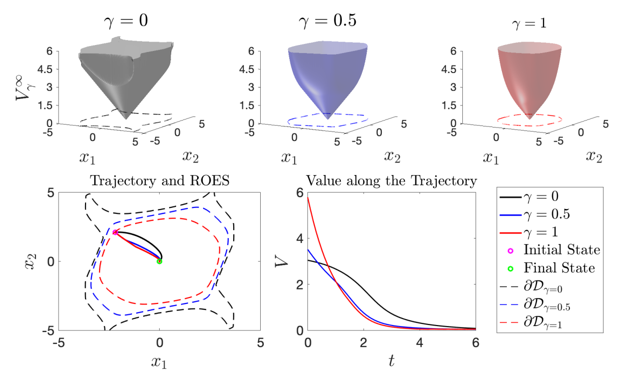

Fig. 1. An illustrative example of the CLVF approach. First row: given a system (39) with control bounds, for different desired exponentially stabilizing rates γ, the computed CLVFs and corresponding regions of exponential stabilizability (ROES) $\mathcal{D}_\gamma$'s are shown. It can be seen that the CLVFs are not smooth and not in quadratic form. When $\gamma = 0$, the ROES of CLVF is its largest control invariant set, and when $\gamma > 0$, the ROES contains all the states that can be stabilized to the origin with exponential rate γ. A larger γ results in smaller $\mathcal{D}_\gamma$. Second row: (left) trajectories using the CLVF-QP controller starting from the same initial condition, (right) the decay of the value along these trajectories. A non-zero γ forces the value to decay; larger γ results in a faster decrease. In this example, the dynamics itself is attractive, and drives the trajectory of $\gamma = 0$ to the origin.

Recent work has shown that CBF-like functions can be constructed by modifying HJ reachability analysis [20]. In this paper we seek to extend this work to constructing control Lyapunov-value functions (CLVFs), which have similar stabilizing properties to CLFs. This extension is nontrivial: HJ reachability applied to liveness traditionally seeks to find the minimum time to reach a goal (and may be forced to exit after reaching it), whereas a CLVF seeks to stabilize to a goal. Additionally, there are many systems for which a valid CLF does not exist due to no stabilizable equilibrium points. For such systems, our method can find and stabilize to the smallest control invariant set around a point of interest.

In this paper, the main contributions are:

1) We define the CLVF and establish the theoretical foundation of how to compute this function.
2) We establish the relation between CLVFs and the exponential stabilizability of nonlinear systems. We provide a numerical estimation of the region of exponential stabilizability (ROES), and demonstrate the effect of the exponential rate parameter γ on the ROES.
3) For systems that have no stabilizable equilibrium points, we show that the CLVF stabilizes the system to its smallest control invariant set around a point of interest (if one exists).
4) We provide a QP-based controller, and show that both feasibility and the exponential decay rate are guaranteed.

II. Background

A. Problem Formulation

Consider the general nonlinear time-invariant system

$$\dot{x}(s) = f\left(x(s), u(s)\right), \quad s \in [t, t_f], \quad x(t) = x, \tag{1}$$

where $t < 0$ is the initial time, $t_f \geq t$, and $x \in \mathbb{R}^n$ is the initial state. The control input u is drawn from a compact set $\in \mathcal{U} \subset \mathbb{R}^m$, and the control function $u(\cdot)$ is assumed to be drawn from the set of measurable functions $\mathbb{U}$. Assuming that the dynamics $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$ is Lipschitz continuous in (x, u) and continuous in the s, there exists a unique solution $\varphi(s) = \varphi(s; x, t, u(\cdot)) : [t, t_f] \to \mathbb{R}^n$ of the system (1) given initial state x and control signal $u(\cdot)$. In this paper, we seek to exponentially stabilize the system to its smallest control invariant set.

Definition 1. A set $\mathcal{I}(t)$ is finite-time control invariant if $\forall x \in \mathcal{I}$, there exists some control $u(\cdot)$ such that $\forall s \in [t, 0]$, $\varphi(s; x, u, t) \in \mathcal{I}$. A set $\mathcal{I}^\infty$ is infinite-time control invariant if the above condition holds as $t \to -\infty$.

Definition 2. The ROES of $\mathcal{I}_m^\infty$ is defined as

$$\mathcal{D}_{\text{ROES}} := \{x \in \mathbb{R}^n | \exists u \in \mathbb{U}, \gamma > 0 \text{ s.t. } ||\varphi(s; x, u, t)|| \to \mathcal{I}_m^\infty \text{ with exponential rate } \gamma \text{ as } t \to -\infty\}.$$

For systems with an stabilizable equilibrium point, the equilibrium point itself is the smallest control invariant set. In this case, without loss of generality, we assume it is $(x, u) = (0, 0)$. For systems that have no equilibrium point, but do have some control invariant sets, without loss of generality, we assume this set is around the origin. For simplicity, in both cases, this set is denoted as $\mathcal{I}_m^\infty$. Throughout the paper, $\varphi(\cdot)$ and $u(\cdot)$ denotes the trajectory function and control function. $\varphi(s)$ refers to the function value at time s.

B. Optimal Control and HJ Reachability

In this paper we introduce the safety (rather than liveness) formulation of Hamilton-Jacobi (HJ) reachability. This is because in this paper we construct the *liveness* problem of stabilizing to the smallest control invariant set $\mathcal{I}_m^\infty$ as a *safety* problem, wherein the system seeks to avoid all regions of the state space that are not $\mathcal{I}_m^\infty$.

To compute the value function for HJ reachability we define a Lipschitz continuous cost function $\ell : \mathbb{R}^n \to \mathbb{R}$ whose super-zero level set is the failure set $\mathcal{F} = \{x : \ell(x) \geq 0\}$. The finite-time horizon cost function captures whether a given trajectory enters $\mathcal{F}$ at any point in the time horizon (conventionally $[t, 0]$):

$$J(t, x, u) = \sup_{s \in [t,0]} \ell(\varphi(s; x, u, t)). \tag{2}$$

The value function is this cost given optimal control:

$$V(x, t) = \inf_{u \in \mathbb{U}_{[t,0]}} J(t, x, u) = \inf_{u \in \mathbb{U}_{[t,0]}} \sup_{s \in [t,0]} \ell(\varphi(s)). \tag{3}$$

This value function is the unique Lipschitz continuous viscosity solution to the following Hamilton-Jacobi-Isaacs variational inequality (HJI-VI) [21]:

$$\min\Big\{\ell(x) - V(x, t), \\ D_t V(x, t) + \inf_{u \in \mathbb{U}_{[t,0]}} D_x V(x, t) \cdot f(x, u)\Big\} = 0. \tag{4}$$

Therefore the value function can be computed using dynamic programming by applying this HJI-VI recursively over time. The infinite-time horizon value function is defined by taking the limit of $V(x, t)$ as $t \to -\infty$ [22],

$$V^\infty(x) = \lim_{t \to -\infty} V(x, t). \tag{5}$$

For the time-varying value function, trajectories that start in the super-zero level set $V(x, t) \geq 0$ will enter $\mathcal{F}$ for some time $s \in [t, 0]$. The sub-zero level set of $V(x, t)$ is therefore safe for the time horizon. This can be extended to say that each sub-α level set $\mathcal{V}_\alpha = \{x : V(x, t) \leq \alpha\}$ is safe with respect to the set defined by $\mathcal{F}_\alpha = \{x : \ell(x) \leq \alpha\}$. In the infinite-time setting, this means every sub-α level set of $V^\infty(x)$ is control invariant and can maintain trajectories within a particular level set boundary. However, since the set is only control invariant, there is no guarantee that the system can be stabilized to lower level sets or the origin.

C. Control Lyapunov Functions

CLFs are a common tool for ensuring and enforcing the stability of a system. A continuous function $V_{\text{clf}} : \mathbb{R}^n \to \mathbb{R}$ is a local CLF for the equilibrium point x if, in a neighborhood $\mathcal{O}$ of x, the following holds: (a) V_{clf} is proper at x, (b) V_{clf} is positive definite on $\mathcal{O}$ and continuously differentiable on $\mathcal{O}$, and (c) for each $x \in \mathcal{O}$, there exists some $u \in \mathcal{U}$ such that $\dot{V}_{\text{clf}}(x) = \frac{dV_{\text{clf}}}{dx} \cdot f(x, u) < 0$.

The existence of a CLF implies the system is asymptotically stabilizable, and the gradient of the CLF can be used to generate a stabilizing controller. If in addition $\dot{V} \leq \gamma V$ for some $\gamma > 0$, exponential stabilizability is guaranteed.

For a local CLF, each sub-level set contained in $\mathcal{O}$ is not only control invariant, but also attractive. Informally speaking, control invariance only guarantees not leaving the control invariant set, whereas attractive guarantees shrinking to a smaller set. This appealing property differs from the sub-level set of the infinite-time value function defined in HJ reachability analysis (5). In this paper we seek to modify (5) to obtain this property.

Note that many nonlinear systems fail to have a continuously differentiable CLF, though there has been work to generalize CLFs to only be continuous based on generalized gradients [23], [24].

D. Control Barrier-Value Functions

Control barrier functions are similar to CLFs but generated for safety problems [4]. Recent work [20] introduced a method of modifying HJ reachability value functions to have similar properties to CBFs. The resulting control barrier value function (CBVF) is the unique Lipschitz continuous viscosity solution to a specific variational inequality called CBVF-VI, and has the property $\dot{B}_\gamma \geq -\gamma B_\gamma$. The CBVF gives the largest control invariant set, and the optimal control is less conservative compared to classic HJ reachability [20].

We emphasize some key differences between the CBVF and our proposed CLVF: 1) The CBVF focuses on staying within the safe set in finite time, while we focus on stabilizing to the goal over an infinite-time horizon. 2) The CBVF is defined on $\mathbb{R}^n$, as is our time-varying CLVF (TV-CLVF), but the infinite-time CLVF is not guaranteed to be defined on $\mathbb{R}^n$.

3) We prove the CLVF is the Lipschitz continuous viscosity solution to the CLVF-VI, whereas the CBVF is only proved for time-varying case. 4) We propose the CLVF-QP which is guaranteed to be feasible for all time, while the CBVF-QP can only guarantee feasibility for a finite-time.

III. Control Lyapunov-Value functions

In this section we introduce the CLVF and its properties.

A. Definition of CLVFs

Definition 3. Time-Varying Control Lyapunov-Value Function (TV-CLVF) A TV-CLVF $V_\gamma(x,t) : \mathbb{R}^n \times \mathbb{R}_- \to \mathbb{R}_+$, is defined as

$$V_\gamma(x,t) = \inf_{u\in\mathbb{U}_{[t,0]}} \sup_{s\in[t,0]} e^{\gamma(s-t)}\ell(\varphi(s;x,t,u)), \tag{6}$$

where $J_\gamma(t,x,u) = \sup_{s\in[t,0]} e^{\gamma(s-t)}\ell(\varphi(s;x,t,u))$ is the finite-time cost, γ is a user specified parameter which represents the desired decay rate, $\ell(x) = ||x||_2 - a$, and a is a constant determined by Algorithm 1. The control seeks to decrease the system along $\ell(x)$ towards the origin. Note that the specification of $\ell(x)$ does not restrict the TV-CLVF to be in quadratic form.

The standard HJ reachability value function (3) is a special case of (6) where $\gamma = 0$. We will show how the inclusion of this exponential term provides the region for which the system can be stabilized at a rate of γ.

Note that since $\forall s \in [t,0]$ and $\gamma \le 0$, $e^{\gamma(s-t)} \ge 1$, we have $V_\gamma(x,t) \ge V_0(x,t)$. Therefore each level set of $V_\gamma(x,t)$ is a subset of the level set of $V_0(x,t)$.

We extend the definition of the TV-CLVF to infinite time:

Definition 4. CLVF Given a compact set D_γ, the function $V_\gamma^\infty : \mathcal{D}_\gamma \subset \mathbb{R}^n \to \mathbb{R}_+$ is a CLVF if the following limit exists:

$$V_\gamma^\infty(x) = \lim_{t\to-\infty} V_\gamma(x,t). \tag{7}$$

The existence of the limit in (7) implies that the CLVF is unique given a specific system and input bounds. The Lipschitz continuity of the CLVF is proved in Theorem 2.

Remark 1. Note that the existence of the limit to negative infinity in (7) is equivalent to $\lim_{n\to\infty} V_\gamma(x,t_n) = V_\gamma^\infty(x)$ for every sequence $\{t_n\}$ s.t. $t_n \neq -\infty$ and $\lim_{n\to\infty} t_n = -\infty$. In other words, if we evaluate $V_\gamma(x,t)$ at each t_n, and get a sequence of functions $\{U_n\}$ such that $U_n(x) = V_\gamma^\infty(x,t_n)$, then $\{U_n\}$ converges to V_γ^∞ pointwise in $\mathcal{D}_\gamma$.

If in addition, the sequence $\{t_n\}$ is monotonically decreasing, by Dini's Theorem, the convergence of $\{U_n\}$ to V_γ^∞ is uniform in $\mathcal{D}_\gamma$.

B. Dynamic Programming Principle & Viscosity Solution

We next establish the theoretical foundation for computing the CLVF in two steps. First, we show that Bellman's optimality principle can be used to derive the dynamic programming principle for the CLVF. Second, we show that the CLVF is the viscosity solution to the CLVF-VI.

Theorem 1. (CLVF Dynamic Programming Principle) For all $t_1 > 0$, the following is satisfied

$$V_\gamma^\infty(x) = \inf_{u\in\mathbb{U}} \max\left\{ \max_{s\in[0,t_1]} e^{\gamma t_1}\ell(\varphi(s;x,0,u)),\ e^{\gamma t_1} V_\gamma^\infty(z) \right\}. \tag{8}$$

We first show that the CLVF can be expressed in an equivalent form.

Proposition 1. An equivalent TV-CLVF $V_{\gamma,f} : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is defined as

$$V_{\gamma,f}(x,t) = \inf_{u\in\mathbb{U}_{[0,t]}} J_{\gamma,f}(t,x,u), \tag{9}$$

with cost $J_{\gamma,f}(t,x,u) = \sup_{s\in[0,t]} e^{\gamma s}\ell(\varphi(s;x,0,u))$. The forward infinite-time cost and the CLVF is defined as

$$J^\infty_{\gamma,f}(x,u) = \lim_{t\to\infty} J_{\gamma,f}(t,x,u)$$
$$V^\infty_{\gamma,f}(x) = \lim_{t\to\infty} V_{\gamma,f}(x,t). \tag{10}$$

We have

$$V^\infty_{\gamma,f}(x) = V^\infty_\gamma(x). \tag{11}$$

Proof. We first show that $\forall t \ge 0$, $J_{\gamma,f}(t,x,u(\cdot)) = J_\gamma(-t,x,\bar{u}(\cdot))$, if $u(\tau-t) = \bar{u}(\tau)$. Since the system is time-invariant, we have

$$\varphi_1(s;x,0,u(\tau)) = \varphi_2(s-t;x,-t,u(\tau-t)),$$
$$\ell\big(\varphi_1(s;x,0,u(\tau))\big) = \ell\big(\varphi_2(s-t;x,-t,u(\tau-t))\big).$$

We multiply by an exponential term and take the supremum:

$$\sup_{s\in[0,t]} e^{\gamma s}\ell\big(\varphi_1(s;x,0,u(\tau))\big)$$
$$= \sup_{s\in[0,t]} e^{\gamma s}\ell\big(\varphi_2(s-t;x,-t,u(\tau-t))\big). \tag{12}$$

The L.H.S. of (12) is

$$\sup_{s\in[0,t]} e^{\gamma s}\ell\big(\varphi_1(s;x,0,u(\cdot))\big) = J_{\gamma,f}(t,x,u(\cdot)).$$

Let $s_1 = s-t$, the R.H.S. of (12) becomes

$$\sup_{s_1\in[-t,0]} e^{\gamma(s_1+t)}\ell(\varphi_2(s_1;x,-t,u(\tau-t))) = J_\gamma(-t,x,\bar{u}(\cdot)),$$

where $u(\tau-t) = \bar{u}(\tau)$. This means $\forall u(\cdot) \in \mathbb{U}_{[0,t]}$, $\exists \bar{u}(\cdot) \in \mathbb{U}_{[-t,0]}$, such that $J_{\gamma,f}(t,x,u(\cdot)) = J_\gamma(-t,x,\bar{u}(\cdot))$.

Assume $u^*(\cdot) \in \mathbb{U}_{[0,t]}$ is the optimal control signal for $V_{\gamma,f}(x,t)$, then $\bar{u}^*(\tau) = u^*(\tau-t)$ is the optimal control signal for $V_\gamma(x,-t)$, i.e.

$$V_{\gamma,f}(x,t) = V_\gamma(x,-t). \tag{13}$$

This can be proved by contradiction. Assume $u^*(\cdot) \in \mathbb{U}_{[0,t]}$ is the optimal control signal for $V_{\gamma,f}(x,t)$, and $\bar{u}^*(\tau) = u^*(\tau-t)$ is **not** the optimal control signal for $V_\gamma(x,-t)$. This means there exist another control signal $\bar{u}_2(\cdot)$, s.t.

$$J_\gamma(-t,x,\bar{u}_2(\cdot)) < J_\gamma(-t,x,\bar{u}^*(\cdot)),$$

then, there exists a control signal $u_2(\tau-t) = \bar{u}_2(\tau)$ s.t.

$$\begin{aligned} J_{\gamma,f}(t,x,u_2(\cdot)) &= J_\gamma(-t,x,\bar{u}_2(\cdot)) \\ &< J_\gamma(-t,x,\bar{u}^*(\cdot)) \\ &= J_{\gamma,f}(t,x,u^*(\cdot)) = V_{\gamma,f}(x,t), \end{aligned}$$

which is a contradiction to the assumption that $u^*(\cdot)$ is the optimal control signal. Since (13) holds for all $t \ge 0$, we can take limit $t \to \infty$:

$$\begin{aligned} V^\infty_{\gamma,f}(x) &= \lim_{t\to\infty} V_{\gamma,f}(x,t) \\ &= \lim_{t\to\infty} V_\gamma(x,-t) = V^\infty_\gamma(x). \end{aligned}$$

□

Proof of Theorem 1. Now, we use $V^\infty_{\gamma,f}$ to derive the dynamic program principle. Let $W(x)$ denote the R.H.S. of equation (8), we first show that $V^\infty_{\gamma,f}(x) \le W(x)$.

For $\forall u(\cdot) \in \mathbb{U}$, $t > 0$, let $z = \varphi(t;x,0,u(\tau))$, $\forall \varepsilon_0 > 0$, $\exists \bar{u} \in \mathbb{U}$ such that

$$V^\infty_{\gamma,f}(z) \ge J^\infty_{\gamma,f}(z,\bar{u}) - \varepsilon_0. \tag{14}$$

Define the control input $\hat{u}(\cdot)$ as,

$$\hat{u}(s) := \begin{cases} u(s) & \text{if} \quad 0 \le s < t, \\ \bar{u}(s-t) & \text{if} \quad s \ge t. \end{cases}$$

From the definition of CLVF, we have

$$V^{\infty}_{\gamma,f}(x) \le J^{\infty}_{\gamma,f}(x,\hat{u}) = \sup_{s\in[0,+\infty)} e^{\gamma s}\ell(\varphi(s;x,\hat{u}(\tau))). \tag{15}$$

The cost $J^{\infty}_{\gamma,f}(x,\hat{u})$ can be written as

$$J^{\infty}_{\gamma,f}(x,\hat{u}) = \max\Big\{ \sup_{s\in[0,t]} e^{\gamma s}\ell\big(\varphi(s;x,0,\hat{u}(\tau))\big), \sup_{s\in[t,+\infty)} e^{\gamma s}\ell\big(\varphi(s;z,t,\hat{u}(\tau))\big)\Big\},$$

which is equivalent to

$$J^{\infty}_{\gamma,f}(x,\hat{u}) = \max\Big\{ \sup_{s\in[0,t]} e^{\gamma s}\ell\big(\varphi(s;x,0,\hat{u}(\tau))\big), e^{\gamma t}\sup_{s\in[0,+\infty)} e^{\gamma s}\ell\big(\varphi(s;z,0,\bar{u}(\tau))\big)\Big\}. \tag{16}$$

By the definition of $J^{\infty}_{\gamma,f}(z,\bar{u})$, combining (15) and (16), for $\forall u \in \mathcal{U}_t$,

$$V^{\infty}_{\gamma,f}(x) \le \max\Big\{ \sup_{s\in[0,t]} e^{\gamma s}\ell\big(\varphi(s;x,0,u(\tau))\big), e^{\gamma t}J^{\infty}_{\gamma,f}(z,\bar{u})\Big\}. \tag{17}$$

From (14) and (17),

$$\begin{aligned} V^{\infty}_{\gamma,f}(x) &\le \max\Big\{ \sup_{s\in[0,t]} e^{\gamma s}\ell\big(\varphi(s;x,0,u(\tau))\big), e^{\gamma t}(V(z)+\varepsilon_0)\Big\} \\ &\le \max\Big\{ \sup_{s\in[0,t]} e^{\gamma s}\ell\big(\varphi(s;x,0,u(\tau))\big), e^{\gamma t}V^{\infty}_{\gamma,f}(z)\Big\} + e^{\gamma t}\varepsilon_0. \end{aligned}$$

By taking $\varepsilon_0 = e^{-\gamma t}\varepsilon$, we get

$$V^{\infty}_{\gamma,f}(x) \le W(x) + \varepsilon, \quad \forall \varepsilon > 0. \tag{18}$$

Since ε and control u are arbitrary, $V^{\infty}_{\gamma,f}(x) \le W(x)$.

For the opposite inequality, by the definition of $W(x)$, for $\forall u \in \mathbb{U}$, it holds that

$$W(x) \le \max\Big\{ \sup_{s\in[0,t]} e^{\gamma s}\ell\big(\varphi(s;,x,0,u(\tau))\big), e^{\gamma t}V^{\infty}_{\gamma,f}(z)\Big\}, \tag{19}$$

here $z = \varphi(t;x,0,u(\tau))$. Take $u_1(\tau+t) := u(\tau)$, by the definition of $V^{\infty}_{\gamma,f}(z)$, we have

$$V^{\infty}_{\gamma,f}(z) \le \sup_{s\in[0,+\infty)} e^{\gamma s}\ell\big(\varphi(s;z,0,u_1(\tau+t))\big). \tag{20}$$

From (19) and (20),

$$\begin{aligned} W(x) &\le \max\Big\{ \sup_{s\in[0,t]} e^{\gamma s}\ell\big(\varphi(s;x,0,u(\tau))\big), e^{\gamma t}\sup_{s\in[0,+\infty)} e^{\gamma s}\ell(\varphi(s;z,0,u_1(\tau+t)))\Big\} \\ &= \max\Big\{ \sup_{s\in[0,t]} e^{\gamma s}\ell\big(\varphi(s;x,0,u(\tau))\big), \sup_{s\in[t,+\infty)} e^{\gamma s}\ell\big(\varphi(s;z,t,u(\tau))\big)\Big\} = J^{\infty}_{\gamma,f}(x,u). \end{aligned}$$

Since u is the arbitrary control, taking the infimum of $J^{\infty}_{\gamma,f}(x,u)$ over $\mathbb{U}$, it still holds that

$$W(x) \le V^{\infty}_{\gamma,f}(x). \tag{21}$$

From (18), (21), and (11) we have $W(x) = V^{\infty}_{\gamma,f}(x) = V^{\infty}_{\gamma}(x)$, so the equality (8) holds. □

Theorem 2. (CLVF-VI viscosity solution) The CLVF is the unique Lipschitz solution to the following CLVF-VI in the viscosity sense,

$$\max\Big\{\ell(x) - V^{\infty}_{\gamma}(x),\ \inf_{u\in\mathbb{U}} D_x V^{\infty}_{\gamma}\cdot f(x,u) + \gamma V^{\infty}_{\gamma}(x)\Big\} = 0. \tag{22}$$

Proof. Following from [13]–[15], analogously a continuous function V^{∞}_{γ} is the viscosity solution to the CLVF-VI if the following statements hold,

(i) V^{∞}_{γ} is a viscosity subsolution of CLVF-VI if for $\forall x \in \mathbb{R}^n$, $\forall p \in D^+V^{\infty}_{\gamma}(x)$,
$$\max\Big\{\ell(x) - V^{\infty}_{\gamma}(x), \inf_{u\in\mathbb{U}} p^+\cdot f(x,u) + \gamma V^{\infty}_{\gamma}(x)\Big\} \ge 0.$$

(ii) V^{∞}_{γ} is a viscosity supersolution of CLVF-VI if for $\forall x \in \mathbb{R}^n$, $\forall p \in D^-V^{\infty}_{\gamma}(x)$,
$$\max\Big\{\ell(x) - V^{\infty}_{\gamma}(x), \inf_{u\in\mathbb{U}} p^-\cdot f(x,u) + \gamma V^{\infty}_{\gamma}(x)\Big\} \le 0.$$

An equivalent definition could be stated in terms of test functions,

(iii) statement (i) holds if and only if, for any $\psi \in C^1(\mathbb{R}^n)$, if x is a local maximum for $V^{\infty}_{\gamma} - \psi$, then
$$\max\Big\{\ell(x) - V^{\infty}_{\gamma}(x), \inf_{u\in\mathbb{U}} D_x\psi(x)\cdot f(x,u) + \gamma V^{\infty}_{\gamma}(x)\Big\} \ge 0.$$

(iv) statement (ii) holds if and only if, for any $\psi \in C^1(\mathbb{R}^n)$, if x is a local minimum for $V^{\infty}_{\gamma} - \psi$, then
$$\max\Big\{\ell(x) - V^{\infty}_{\gamma}(x), \inf_{u\in\mathbb{U}} D_x\psi(x)\cdot f(x,u) + \gamma V^{\infty}_{\gamma}(x)\Big\} \le 0.$$

We first prove that $V^{\infty}_{\gamma}(x)$ is a viscosity subsolution. Let $\psi \in \mathbb{C}^1(\mathbb{R}^n)$ and x be a local maximum of $V^{\infty}_{\gamma} - \psi$. Then for some $r > 0$, it holds that

$$V^{\infty}_{\gamma}(x) - \psi(x) \ge V^{\infty}_{\gamma}(z) - \psi(z), \tag{23}$$

for $\forall z \in B(x,r)$. Suppose there exists sufficiently small t_0 such that for any arbitrary control input $u \in \mathbb{U}$,

$$\varphi(t;x,0,u) \in B(x,r), \qquad \forall t \in [0,t_0].$$

By the DPP (8), $V(x)$ could be taken from two cases, so we first assume that

$$V^{\infty}_{\gamma}(x) = V^{\infty}_{\gamma,f}(x) = \inf_{u\in\mathbb{U}} e^{\gamma t}\cdot V^{\infty}_{\gamma,f}(\varphi(t;x,0,u)),$$

then there exists $u^* \in \mathbb{U}$ such that

$$V^{\infty}_{\gamma}(x) = e^{\gamma t}\cdot V^{\infty}_{\gamma,f}(\varphi(t;x,0,u^*)).$$

Combining the equation above with (23), and let $z = \varphi(t;x,0,u)$, it holds that

$$\begin{aligned} &\psi(x) - \psi(\varphi(t;x,0,u)) \\ &\le e^{\gamma t}V^{\infty}_{\gamma,f}(\varphi(t;x,0,u^*)) - V^{\infty}_{\gamma,f}(\varphi(t;x,0,u)) \\ &\le (e^{\gamma t}-1)V^{\infty}_{\gamma,f}(\varphi(t;x,0,u)). \end{aligned} \tag{24}$$

Divide by $t > 0$ on the L.H.S. and the last inequality of (24), and let $t \to 0$. By the differentiability of ψ and the continuity of f and φ, the L.H.S. becomes

$$\lim_{t\to 0} \frac{\psi(x) - \psi(\varphi(t;x,0,u))}{t} = -D_x\psi(x)\cdot f(x,u).$$

Similarly, by the continuity of φ, the last inequality of (24) becomes,

$$\lim_{t\to 0} V^{\infty}_{\gamma,f}(\varphi(t;x,0,u))\frac{e^{\gamma t}-1}{t} = \gamma V^{\infty}_{\gamma,f}(x) = \gamma V^{\infty}_{\gamma}(x).$$

Therefore, (24) becomes,

$$D_x\psi(x)\cdot f(x,u) + \gamma V^{\infty}_{\gamma}(x) \ge 0.$$

Since the control u is arbitrary, we have

$$\inf_{u\in\mathbb{U}} D_x\psi(x)\cdot f(x,u) + \gamma V^{\infty}_{\gamma}(x) \ge 0. \tag{25}$$

For the other case, we assume that,

$$V^{\infty}_{\gamma}(x) = V^{\infty}_{\gamma,f}(x) = \inf_{u\in\mathbb{U}} \sup_{s\in[0,t]} e^{\gamma s}\ell(\varphi(s;x,0,u)),$$

and there exists u^* such that

$$V_\gamma^\infty(x) = \sup_{s\in[0,t]} e^{\gamma s}\ell(\varphi(s;x,0,u^*)). \tag{26}$$

From (23) and (26),

$$\begin{aligned}\psi(x) - \psi(\varphi(t;x,0,u)) &\leq \sup_{s\in[0,t]} e^{\gamma s}\ell(\varphi(s;x,0,u^*)) - V_\gamma^\infty(\varphi(s;x,0,u)) \\ &\leq \sup_{s\in[0,t]} e^{\gamma s}\ell(\varphi(s;x,0,u)) - V_\gamma^\infty(\varphi(s;x,0,u)).\end{aligned}$$

Since the equation above holds for $\forall t \in [0, t_0]$, take $t \to 0$ and by the continuity of ψ, ℓ and φ, it holds that

$$0 \leq \ell(x) - V_\gamma^\infty(x). \tag{27}$$

Combining (25) and (27), it holds that

$$\max\Big\{\ell(x) - V_\gamma^\infty(x), \inf_{u\in\mathbb{U}} D_x\psi(x)\cdot f(x,u) + \gamma V(x)\Big\} \geq 0. \tag{28}$$

Thus, $V_\gamma^\infty(x)$ is a viscosity subsolution of CLVF-VI.

Next we prove that $V_\gamma^\infty(x)$ is also a viscosity supersolution of CLVF-VI. Assume that x is local minimum of $V_\gamma^\infty - \psi$, then for some $r > 0$,

$$V_\gamma^\infty(x) - \psi(x) \leq V_\gamma^\infty(z) - \psi(z), \qquad \forall z \in B(x,r). \tag{29}$$

Suppose for a sufficiently small t, we first assume that,

$$V_\gamma^\infty(x) = V_{\gamma,f}^\infty(x) = \inf_{u\in\mathbb{U}} e^{\gamma t} V_{\gamma,f}^\infty(\varphi(t;x,0,u)).$$

Then for an arbitrary $\varepsilon > 0$, $\exists \bar{u} \in \mathbb{U}$ s.t.

$$V_\gamma^\infty(x) \geq e^{\gamma t} V_\gamma^\infty(\varphi(t;x,0,\bar{u})) - t\varepsilon. \tag{30}$$

Let $z = \varphi(t;x,0,\bar{u})$, from (29) and (30),

$$\psi(x) - \psi(\varphi(t;x,0,\bar{u})) \geq (e^{\gamma t} - 1)V_\gamma^\infty(\psi(t;x,0,\bar{u})) - t\varepsilon.$$

For the above equation, divide by $t > 0$ and pass the limit to 0, we see $-D_x\psi \cdot f(x,\bar{u}) \geq \gamma V_\gamma^\infty(x) - \varepsilon$. The term from the L.H.S. could be estimated from above.

Equivalently, $\inf_{u\in\mathbb{U}} D_x\psi \cdot f(x,u) + \gamma V_\gamma^\infty(x) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary,

$$\inf_{u\in\mathbb{U}} D_x\psi \cdot f(x,u) + \gamma V_\gamma^\infty(x) \leq 0. \tag{31}$$

Next, suppose that

$$V_\gamma^\infty(x) = V_{\gamma,f}^\infty(x) = \inf_{u\in\mathbb{U}} \sup_{s\in[0,t]} e^{\gamma s}\ell(\varphi(s;x,0,u)).$$

Then for $\forall \varepsilon > 0$, $\exists \bar{u} \in \mathbb{U}$ s.t.

$$V_\gamma^\infty(x) \geq \sup_{s\in[0,t]} e^{\gamma s}\ell(\varphi(s;x,0,\bar{u})) - \varepsilon. \tag{32}$$

Let $z = \varphi(s;x,0,\bar{u})$, combine (29) and (32),

$$\begin{aligned}&\psi(x) - \psi(\varphi(s;x,0,\bar{u})) \\ &\quad \geq \sup_{s\in[0,t]} e^{\gamma s}\ell(\varphi(s;x,0,\bar{u})) - V_\gamma^\infty(\varphi(s;x,0,\bar{u})) - \varepsilon\end{aligned}$$

Let $t \to 0$, then $s \to 0$, by the continuity of φ, ψ and ℓ, it follows that $\ell(x) - V_\gamma^\infty(x) \leq \varepsilon$. Since ε is arbitrary,

$$\ell(x) - V_\gamma^\infty(x) \leq 0 \tag{33}$$

From (31) and (33), we conclude that $V_\gamma^\infty(x)$ is a viscosity supersolution of CLVF-VI.

The Lipschitz continuity can be proved following the same process as in [12, Thrm 3.2]. Combined, we proved $V_\gamma^\infty(x)$ is the Lipschitz continuous viscosity solution to (22). □

Note the differences between (22) and (4) are: first, (22) is time-invariant and second, the $\gamma V_\gamma^\infty(x)$ in (22).

C. Algorithm for Computing CLVF

Here we present an Algorithm to compute the CLVF. Specifically, we emphasize how the constant a in $\ell(x)$ is determined. Remark 2 and Proposition 2 are vital.

Remark 2. In classic HJ reachability, an interesting observation is that

$$\begin{aligned}V(x,t) + c &= \inf_{u\in\mathcal{U}_{[t,0]}} \sup_{s\in[t,0]} \big\{\ell(\varphi(s))\big\} + c \\ &= \inf_{u\in\mathcal{U}_{[t,0]}} \sup_{s\in[t,0]} \big\{\ell(\varphi(s)) + c\big\}.\end{aligned}$$

This means for a given system, at any pair (x,t), adding a constant value to the terminal cost $\ell(x)$, the resulting value function will also be added with the same constant. However, this is generally not true for the CLVF.

Proposition 2. For a given system and fixed terminal cost $\ell(x)$, CLVFs with different γ's have the same zero-level set.

Proof. It suffices to prove the following: if $V_{\gamma_1}(x,t) = 0$, then $\forall \gamma \geq 0$, $V_\gamma(x,t) = 0$. It is clear that $e^{\gamma(s-t)} \geq 0$ holds for all $\gamma \geq 0$ and $s \in [t,0]$. If $V_{\gamma_1}(x,t) = 0$, there exist some $u(\cdot)$ such that $\sup_{s\in[t,0]} e^{\gamma_1(s-t)}\ell(x) = 0$. For a different $\gamma_2 > 0$, follow the same control signal $u(\cdot)$, we have $\sup_{s\in[t,0]} e^{\gamma_2(s-t)}\ell(x) = 0$, which proves the proposition. □

Algorithm 1: Obtaining the CLVF for general non-linear systems

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1 **Input**: System model, input bounds, desired exponential rate $\gamma > 0$.
2 **Output**: CLVF $V_\gamma^\infty(x)$.
3 Find the smallest control invariant set: set $\ell_1(x) = ||x||_2$ and $\gamma_1 = 0$, use (8) to get $V^\infty(x)$
4 Compute the CLVF: $a \leftarrow \min_x V^\infty(x)$, set $\ell_2(x) = ||x||_2 - a$, use (8) to get $V_\gamma^\infty(x)$.

The procedure for computing CLVFs is shown in Algorithm 1. Line 3 computes a classic HJ value function with $\gamma = 0$. If this value function exists, the level set of its minimum value is $\mathcal{I}_m^\infty$ of this system given input bounds. Line 4 re-initializes the computation to compute the CLVF. By remark 2, if in Line 4 we again set $\gamma = 0$, the zero level set of the resulting function will be $\mathcal{I}_m^\infty$. By Proposition 2, the zero level set of CLVF with a positive γ will be $\mathcal{I}_m^\infty$. This means the CLVF is positive outside $\mathcal{I}_m^\infty$, and has zero value $\forall x \in \mathcal{I}_m^\infty$.

D. Infinite-Time CLVF and Exponential stabilizability

Here we establish the relationship between the exponential stabilizability to $\mathcal{I}_m^\infty$ and the convergence of the CLVF. We show that the CLVF will converge locally (globally) if and only if $\mathcal{I}_m^\infty$ is locally (globally) exponentially stabilizable. The following proposition will help us establish the result.

Proposition 3. At any point (differentiable or non-differentiable) in the domain $\mathcal{D}_\gamma$ of the CLVF, there exists some control $u \in \mathcal{U}$ such that

$$\dot{V}_\gamma^\infty \leq -\gamma V_\gamma^\infty. \tag{34}$$

Proof. The CLVF-VI (22) guarantees that the above inequality holds at every point where $V_\gamma^\infty(x)$ is differentiable.

However, by theorem 2, the value function is only Lipschitz continuous, which means there exist points that are not differentiable. For those points, [15] showed that either a super-differential ($D^+V_\gamma^\infty(x)$) or a sub-differential ($D^-V_\gamma^\infty(x)$) exists, whose elements are called super-gradients and sub-gradients respectively. A function is differentiable at x if $D^-V_\gamma^\infty(x) = D^+V_\gamma^\infty(x)$. Non-differentiable points only have a super-differential or sub-differential. At non-differentiable points, $\dot{V}_\gamma^\infty(x) = p \cdot f(x,u)$, where p is either a sub-gradient or a super-gradient.

Considering a non-differentiable point with a super-differential, the corresponding solution is called a sub-solution, which satisfies

$$\max\left\{\ell(x) - V_\gamma^\infty(x), \inf_{u\in\mathbb{U}} p^+ \cdot f(x,u) + \gamma V_\gamma^\infty(x)\right\} \leq 0 \quad \forall p^+ \in D^+V_\gamma^\infty(x).$$

In this case, the maximum of the two terms is less or equal to 0, which implies both terms must be less or equal to 0. We have

$$\forall p^+ \in D^+V_\gamma^\infty(x) \ , \ \inf_{u\in\mathbb{U}} p^+ \cdot f(x,u) \leq -\gamma V_\gamma^\infty(x),$$

which means any super-gradients will provide a decrease of the value along the trajectory. When there exists sub-differential, we have:

$$\max\left\{\ell(x) - V_\gamma^\infty(x), \inf_{u\in\mathbb{U}} p^- \cdot f(x,u) + \gamma V_\gamma^\infty(x)\right\} \geq 0. \quad \forall p^- \in D^-V_\gamma^\infty(x).$$

In this case the largest decrease we can have is

$$\forall p^- \in D^-V_\gamma^\infty(x) \ , \ \inf_{u\in\mathbb{U}} p^- \cdot f(x,u) = -\gamma V_\gamma^\infty(x),$$

which is proved to be true in [16]. Combining all the cases discussed above, we get the desired inequality: $\dot{V}_\gamma^\infty \leq -\gamma V_\gamma^\infty$ holds for all points in $\mathcal{D}_\gamma$. □

Theorem 3. The system can be exponentially stabilized to its smallest control invariant set $\mathcal{I}_m$ from $\mathcal{D}_\gamma \setminus \mathcal{I}_m$ (or $\mathbb{R}^n \setminus \mathcal{I}_m$), if and only if the CLVF exists in $\mathcal{D}_\gamma$ (or $\mathbb{R}^n$).

Proof. ($\leftarrow$) Assume the limit in (7) exists in $\mathcal{D}_\gamma$. For any initial state $x_0 \in \mathcal{D}_\gamma \setminus \mathcal{I}_m$, consider the optimal trajectory $\varphi(s) = \varphi(s; x, u^*, t)$ $\forall s \in [t, 0]$. From Proposition 3:

$$D_x V_\gamma^\infty(x) \cdot f(x, u^*) = \dot{V}_\gamma^\infty \leq -\gamma V_\gamma^\infty. \tag{35}$$

Using the comparison principle, we have $\forall s \in [t, 0]$,

$$V_\gamma^\infty(\varphi(s)) \geq e^{-\gamma(s-0)} V_\gamma^\infty(\varphi(0)). \tag{36}$$

Applying the definition of TV-CLVF,

$$V_\gamma(\varphi(0), 0) = \ell(\varphi(0)) = ||\varphi(0)|| - a.$$

Combining the fact that $V_\gamma(x,t) \leq V_\gamma^\infty(x)$, we have:

$$||\varphi(0)|| = V_\gamma(\varphi(0), 0) + a \leq V_\gamma^\infty(\varphi(0)) + a.$$

Applying (36) gives us

$$\begin{aligned}||\varphi(0)|| &\leq V_\gamma^\infty(\varphi(0)) + a \leq e^{\gamma s} V_\gamma^\infty(\varphi(s)) + a \\ &= \frac{e^{-\gamma(0-s)} V_\gamma^\infty(\varphi(s))}{||\varphi(s)||} ||\varphi(s)|| + a.\end{aligned}$$

This inequality holds for all $s \in [t, 0]$, so it holds for $s = t$:

$$\begin{aligned}||\varphi(0)|| &\leq \frac{e^{-\gamma(0-t)} V_\gamma^\infty(\varphi(t))}{||\varphi(t)||} ||\varphi(t)|| + a \\ &= e^{-\gamma(0-t)} k ||\varphi(t)|| + a\end{aligned}$$

where $k = \frac{V_\gamma^\infty(\varphi(t))}{||\varphi(t)||}$. Since the limit in (7) exists, we know that $V_\gamma^\infty(\varphi(t))$ is finite. Additionally, $||\varphi(t)||$ is the distance to the origin of the initial state, which is also finite. This inequality holds for all $\varphi(t) \in \mathcal{D}_\gamma \setminus \mathcal{I}_m^\infty$. Therefore, we conclude that $\forall \varphi(t) \in \mathcal{D}_\gamma \setminus \mathcal{I}_m^\infty$, there exist some constants k and γ such that

$$||\varphi(0)|| \leq e^{-\gamma(0-t)} k ||\varphi(t)|| + a.$$

In other words, the controlled system can be locally exponentially stabilized to $\mathcal{I}_m$. If the limit exists in $\mathbb{R}^n$, we conclude that the above result holds globally.

($\rightarrow$) Again, consider the optimal solution $\varphi(s) = \varphi(s; x, u^*, t)$ $\forall s \in [t, 0]$. Assume exist some constants k and γ and a set $\mathcal{D}_\gamma$ such that $\forall s \in [t, 0]$ and $\forall \varphi(s) \in \mathcal{D}_\gamma$,

$$||\varphi(s)|| - a \leq k ||\varphi(t)|| e^{-\gamma(s-t)}. \tag{37}$$

Then, plugging (37) into (6),

$$\begin{aligned}V_\gamma(x,t) &= \sup_{s\in[t,0]} e^{\gamma(s-t)} \Big(||\varphi(s)|| - a\Big) \\ &\leq \sup_{s\in[t,0]} e^{\gamma(s-t)} (k ||\varphi(t)|| e^{-\gamma(s-t)}) = k ||\varphi(t)||.\end{aligned}$$

As $t \to -\infty$, $V_\gamma(\varphi, t)$ monotonically increases by definition, and is upper bounded by $k||\varphi(t)||$, we conclude that the infinite-time CLVF exists in $\mathcal{D}_\gamma$. If the above conditions hold globally, the infinite-time CLVF exists globally. □

Theorem 3 establishes the equivalence between the existence of CLVF and the exponential stabilizability of the systems. It implies that the $\mathcal{D}_\gamma = \mathcal{D}_{ROES}$.

The main benefits of using CLVF are 1) each sub-level set of CLVF is not only control invariant, but also attractive, and the decay rate is exponential. 2) For systems with control invariant sets, trajectories within $\mathcal{D}_\gamma$ can be stabilized to $\mathcal{I}_m$. 3) The CLVF can be constructed numerically, removing the need for “guess and check” methods.

IV. Feasibility Guaranteed CLVF-QP

In this section, we specify one way of synthesizing the optimal control for control affine systems using quadratic programming. We emphasize that the proposed quadratic program (QP) is guaranteed to be feasible $\forall x \in \mathcal{D}_\gamma$. A control affine system is of the form

$$\dot{x}(s) = f\big(x(s), u(s)\big) = g\big(x(s)\big) + h\big(x(s)\big) u(s), \tag{38}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}^{n\times m}$. For such systems, (34)is equivalent to the following linear inequality in u:

$$D_x V_\gamma^\infty(x) \cdot g(x) + \inf_{u\in\mathcal{U}} D_x V_\gamma^\infty(x) \cdot h(x) u \leq -\gamma V_\gamma^\infty(x).$$

Similarly to the CLF-QP, we synthesize the optimal controller based on a point-wise min-norm optimization problem with one linear inequality constraint. If in addition, the input bound $\mathcal{U}$ is polytopic, the optimization becomes a QP.

Theorem 4. (Feasibility Guaranteed CLVF-QP) Given some reference control u_r, the optimal controller can be synthesized by the following CLVF-QP with guaranteed feasibility $\forall x \in \mathcal{D}_\gamma$.

$$\begin{aligned}\min_{u\in\mathcal{U}} \quad & (u - u_r)^T (u - u_r) \\ \text{s.t.} \quad & D_x V_\gamma^\infty(x) \cdot g(x) + D_x V_\gamma^\infty(x) \cdot h(x) u \leq -\gamma V_\gamma^\infty(x)\end{aligned}$$

Proof. This is a direct result from Proposition 3. □

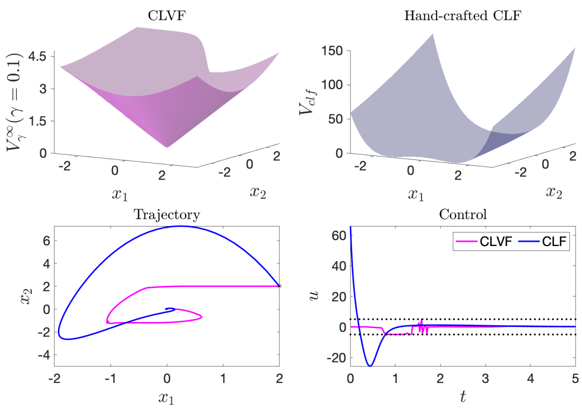

Fig. 2. Comparison of the CLVF (top-left) and the hand-crafted CLF (top-right). The input constraint is $u \in [-5, 5]$, with $\gamma = 1$. Bottom: comparison of trajectories (left) and control signal (right) using back stepping (red) and the CLVF-QP (blue). When using backstepping controller, the control is unbounded, and there is no guarantee of the convergence rate.

Remark 3. Though the QP is feasible, the optimal controller is not necessarily continuous in x. This comes from the non-smooth nature of the CLVF.

V. Numerical Examples

The following examples illustrate the main benefits of our proposed method: feasibility guarantees, input bounds, and applicability to systems with no equilibrium point. We also study the effect γ on the ROES. All simulation results are based on MATLAB and tool boxes [25], [26]. Note that one challenge of the proposed method is how to numerically characterize the region where the limit in (7) does not exist. In the examples provided, we set a very large threshold and treat those states with value larger than the threshold as infinity, hence not in $\mathcal{D}_\gamma$.

A. Effect of γ

Consider the system given by

$$\begin{aligned} \dot{x}_1 &= -\sin(x_1) - 0.5\sin(x_1 - x_2), \\ \dot{x}_2 &= -0.5\sin(x_2) - 0.5\sin(x_2 - x_1) + u, \end{aligned} \tag{39}$$

where $u \in [-0.5, 0.5]$. It can be verified that $(x, u) = (0, 0)$ is an equilibrium point. We compute the CLVF for different γs and provide the estimate (under-approximation) of the corresponding ROESs. The CLVF-QP is used to calculate the optimal trajectories with initial state $x = [-2.2, 2.1]$. Results are shown in Fig. 1. Larger γ's lead to faster stabilization but a smaller ROES, allowing one to trade off aggressiveness of the controller with the region that can be stabilized.

B. Comparison with Back Stepping and Hand-Crafted CLF

Consider the following system

$$\dot{x}_1 = -\frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 - x_2, \quad \dot{x}_2 = u.$$

One can find a global CLF (if no input constraints) using back stepping: $V_{\text{clf}} = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + \frac{3}{2}x_1^2)^2$, and the corresponding controller is: $u(x) = 3x_1(\frac{3}{2}x_1^2 + \frac{1}{2}x_1^3 + x_2) + x_1 - (x_2 + \frac{3}{2}x_1^2)$. Finding this CLF requires some expertise, and may not be valid under input constraints. However, a CLVF with input constraints can be directly computed as shown in Fig. 2. Due to the non-smooth nature of the viscosity solution, the corresponding feedback law is not necessarily continuous.

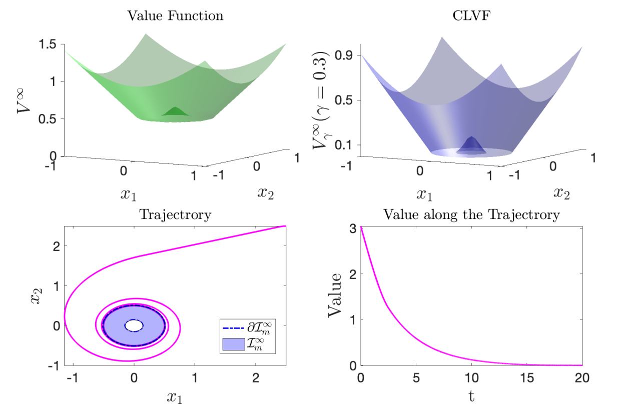

Fig. 3. These plots shows how to use Algorithm 1 to find a positive semi-definite CLVF, and shows the exponential decay rate of the value along the trajectory. Top-left: using step 1 (Line 3), the projected CLVF is shown with $\gamma = 0$. The minimum value is $V_{\min} = 0.51$. Top-right: using step 2 (Line 4), the projected CLVF is shown with $\gamma = 0.3$ and $\ell(x) = ||x||_2 - 0.51$. In this example, we project the third dimension, and take it's minimum value, i.e. $V_{\text{proj}}(x_1, x_2) = \min_{x_3} V(x_1, x_2, x_3)$. Bottom-left: Using the CLVF-QP, the trajectory starting from $x_0 = [2.5, 2.5]$, converges to $\mathcal{I}_m$. Bottom-right: Value of the CLVF along the trajectory.

C. Systems without an Equilibrium Point

Consider the following dubins car example

$$\dot{x}_1 = v\cos(x_3), \quad \dot{x}_2 = v\sin(x_3), \quad \dot{x}_3 = u,$$

where $v = 1$ and $u \in [-\pi, \pi]$. This system cannot be stabilized to a point and therefore does not admit an equilibrium point, nor a CLF. However, our method can identify and stabilize to the smallest region that the system can stay within. The result is shown in Fig. 3, where for visualization the projection through the union of x_3 is shown.

VI. Conclusions

In this paper we proposed a method of finding a Lipschitz continuous control Lyapunov-value function (CLVF) which has properties similar to CLFs and can be computed numerically using dynamic programming. The proposed method can be applied to general nonlinear control systems. For control-affine systems, a feasibility guaranteed CLVF-QP is proposed to synthesize optimal controllers online.

The CLVF approach has two particularly useful properties. First, for systems with no equilibrium points, the proposed method can stabilize the system to the smallest control invariant set (if one exists) around a point of interest. Second, given a desired stabilizability rate γ and control bounds, this method can find the true ROESs. It should be noted that due to numerical issue, the ROESs shown in this paper are under approximation. One can change the parameter γ to trade off the rate under which the system can be stabilized with the region from which the system can be stabilized.

The main drawback to this method is the heavy computation requirement (i.e. "curse of dimensionality") due to dynamic programming. Future work to address this issue includes decomposition of the system [17], using hand-tuned CLF candidates to warm-start the computation [27], adaptive grids [19], and data-driven approaches [18]. Other directions of interest include incorporating disturbances, exploring systems with multiple isolated equilibrium points, finding connections with black-box models, and tuning γ's online to achieve different stabilizability properties.

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